MMP Learning Seminar

Week 45.

Content:

Descending chain condition for volumes.

Birational Automorphisms of Varieties of General Type

(Hacon – McKernan – Xu).

Thm 1.1 $\# \mathrm{Bir}(X) \leq c \cdot \mathrm{vol}(X, K_X)$.

Thm 1.4 (DCC of volume for global quotients)

$\{ \mathrm{vol}(X, K_X+\Delta) : (X,\Delta) \text{ global quotient} \}$ is DCC.

Thm 1.8 (Deformation invariance of log plurigenera)

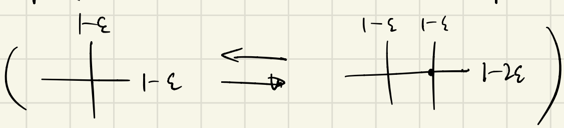
$\pi : X \to T$ proj. morphism of smooth vars. (X,Δ) lc and SNC over T.

(1) If $K_X+\Delta$ is klt, and either $K_X+\Delta$ or Δ is big over T. $m \in \mathbb{Z}_+$ s.t. $m\Delta$ Cartier.

Then $h^0(X_t, m(K_{X_t}+\Delta_t))$ is constant for $t \in T$.

(2) $\mathrm{vol}(X_t, K_{X_t}+\Delta_t)$ is constant for $t \in T$.

Pf (1) Blowing up (X,Δ) to make it a terminal pair.



Apply Prop 4.2.

(2) $\mathrm{vol}(X_t, K_{X_t}+\Delta_t) = \lim_{\varepsilon \to 0} \mathrm{vol}(X_t, K_{X_t}+(1-\varepsilon)\Delta_t)$ $\square$

Thm 1.9 (DCC of vol for bir bounded family)

Fix $I \subseteq [0,1]$ with DCC. $\mathfrak{D}$ a set of SNC pairs (X,Δ) which is log bir bounded, and $\mathrm{coeff}(\Delta) \subseteq I$. Then $\{ \mathrm{vol}(K_X+\Delta) : (X,\Delta) \in \mathfrak{D} \}$ has DCC.

Thm 3.1 (Criteria for log bir bdded family)

Fix $n, A, \delta > 0$. The set of (X, Δ) satisfying:

(1) (X, Δ) lc, proj, dim n

(2) $\mathrm{coeff}(\Delta) \geq \delta$

(3) $\exists m$ s.t. $\mathrm{vol}(m(K_X+\Delta)) \leq A$ and $\phi_{K_X+m(K_X+\Delta)}$ is birational.

is log bir. bounded.

Thm 6.1 ($\phi_{m(K_X+\Delta)}$ is bir for large m) (Tsuji)

(X, Δ) global quotient, $K_X+\Delta$ big. Then $\exists\, C(n)$ s.t. $\phi_{m(K_X+\Delta)}$ is bir if $m \geq C+1$ and $\mathrm{vol}(X, (m-1)(K_X+\Delta)) > (nC)^n$.

"**Lemma 2.3.4**" (Potentially bir divisor $\approx$ bir map)

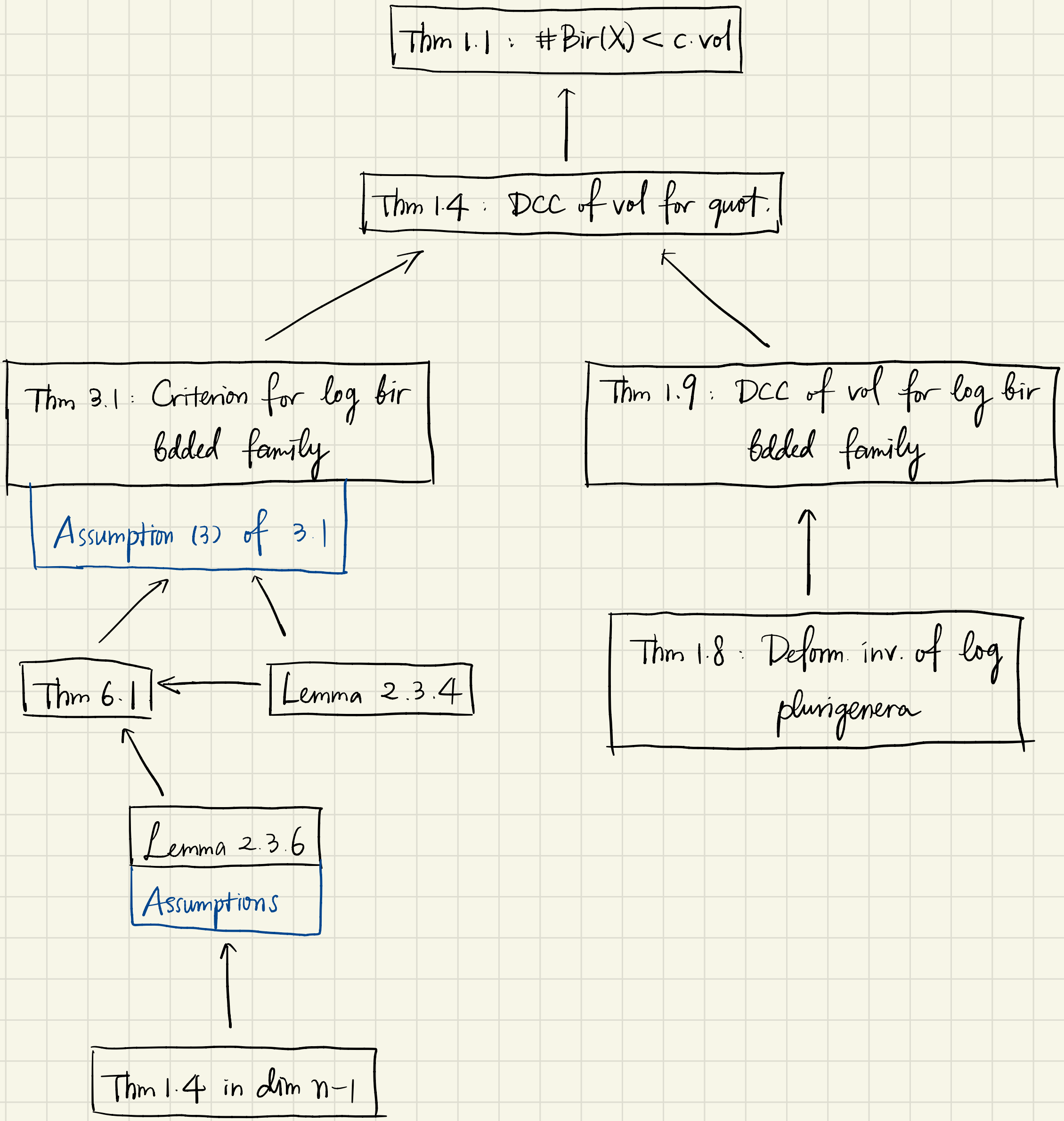
(1) D potentially bir $\Rightarrow$ $\phi_{K_X+\lceil D \rceil}$ bir

(2) ϕ_D bir $\Rightarrow$ $(2n+1)\lfloor D \rfloor$ is potentially bir.

"**Lemma 2.3.6**" (Numerical Criteria for potentially bir divisors)

(* Can be checked on subvar of X $\rightsquigarrow$ induction on dim X).

Structure of Paper



1. Deformation Invariance of Log Plurigenera

Prop 4.1 (Simultaneous log terminal models)

(X,Δ) $\mathbb{Q}$-factorial log pair. T smooth curve.

$(X,\Delta) \longrightarrow T$ proj. morphism s.t. every component of Δ dominates T

and (X_t, Δ_t) terminal, $\forall t \in T$.

$0 \in T$ closed pt. Suppose

(1) Δ or $K_X + \Delta$ big over T

(2) no components of Δ_0 belongs to $\mathrm{Bs}_{\mathbb{R}}(K_{X_0} + \Delta_0)$

Then $\exists$ lt model $(X,\Delta) \overset{f}{\dashrightarrow} Y$ over T s.t.

- f is isom at generic pts of Δ_0
- $X_0 \overset{f_0}{\dashrightarrow} Y_0$ is a weak lt model of (X_0, Δ_0).

Sketch

Proof (Assume $K_X + \Delta$ is psef)

(1) + BCHM $\Rightarrow$ Run MMP $(K_X+\Delta)$-MMP over T:

$$\begin{tikzcd} (X^k, \Delta^k) \arrow[r, dashed, "g^k"] & (X^{k+1}, \Delta^{k+1}) \\ X_0 \arrow[u, hook] \arrow[r, dashed, "g_0^k"] & X_0^{k+1} \arrow[u, hook] \end{tikzcd}$$

Claim: $\forall k$

(a) g^k is an isom at generic pts of Δ_0^k.

(b) (X_0^k, Δ_0^k) is terminal.

(c) g_0^k is a bir contraction. $\square$

Prop 4.2 $X \to T$ flat proj morphism. (X, Δ) log pair.

Every component of Δ dominates T. (X_t, Δ_t) $\mathbb{Q}$-factorial terminal $\forall t \in T$.

For any component R of Δ, $R \to T$ has irred fibers.

Let $m > 1$ s.t. $m(K_X + \Delta) =: D$ is Cartier.

Suppose Δ or $K_X + \Delta$ is big over T, then

$$h^0(X_t, D_t) \text{ is constant for } t \in T.$$

Sketch

Proof

$$\mathcal{O}_X(D - X_t) \to \mathcal{O}_X(D) \to \mathcal{O}_{X_t}(D_t)$$

$$0 \to H^0(D - X_t) \to H^0(D) \to H^0(D_t) \to H^1(D - X_t)$$

Obs: if $H^1(D - X_t) = 0$ $\forall t \in T$, then done.

KV vanishing: require K_X + (big & nef) + (small)

$$D - X_t = m(K_X + \Delta) - X_t.$$

$\square$

2. DCC of vol for bir bdded family

<u>Def</u> (b-divisor)

$\mathcal{B}(v) = a_v$

A b-div $\mathcal{B}$ on X is $\mathcal{B} = \sum_v a_v \cdot v \quad a_v \in \mathbb{R}$

over all val v on X s.t. $\forall$ bir model $Y \dashrightarrow X$,

$\{v : a_v \neq 0$ & v is a div on $Y\}$ is finite.

$\mathcal{B}_Y := \sum a_v \cdot B_v$ (B_v is the div on Y corresp. to v) is a finite sum.

(Equiv, $\mathcal{B}$ is a system of div $\{\mathcal{B}_Y : Y$ bir model of $X\}$

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, dashed] & & Y_2 \arrow[dl, dashed] \\ & X & \end{tikzcd} \qquad f_* \mathcal{B}_{Y_1} = \mathcal{B}_{Y_2} \Big)$$

<u>Def</u> Fix (X, Δ) log pair

$$\mathcal{M}_\Delta(v) = \begin{cases} \operatorname{mult}_B(\Delta) & , \text{ if } \operatorname{center}(v) = B \text{ is a div on } X \\ 1 & , \text{ else.} \end{cases}$$

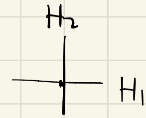
($\mathcal{M}_{\Delta, X} = \Delta$, coeff 1 for any exc. div.)

$\mathcal{L}_\Delta(v) = \max\{0, -a(X, \Delta; v)\}$, i.e.

$Y \xrightarrow{\pi} X$, $\quad K_Y + \Gamma = \pi^*(K_X + \Delta) + E$, $\quad \Gamma, E \geqslant 0$.

then $\mathcal{L}_{\Delta, Y} = \Gamma$.

Ex. $(\overset{X=}{\mathbb{A}^2},\ \overset{\Delta=}{aH_1+bH_2})$



v = weighted blow-up $(v_1, v_2) \in \mathbb{N}^2$

$a(X, \Delta; v) = v_1(1-a) + v_2(1-b) - 1$.

Lemma 5.3 (X, Δ) proj. log pair, X, Y, Z, $\mathbb{Q}$-factorial.

(1) $Y \to X$, then

$$\mathrm{vol}(X, K_X+\Delta) = \mathrm{vol}(Y, K_Y + \mathcal{L}_{\Delta,Y}) = \mathrm{vol}(Y, K_Y+\Gamma)$$

if $\Gamma - \mathcal{L}_{\Delta,Y} \geq 0$ is exceptional.

(2) $X \xrightarrow{\pi} Z$, $\Theta = \Delta \wedge \mathcal{L}_{\pi_*\Delta, X}$, then

$$\mathrm{vol}(K_X+\Delta) = \mathrm{vol}(K_X+\Theta).$$

"$\mathcal{L}_\Delta$ is the smallest divisor s.t. vol is unchanged."

Lemma 5.4 / Notation

(Z, Φ) SNC, lc pair. Suppose $\mathcal{L}_\Phi(v) > 0$, then

center$(v) = W$ is a stratum of (Z, Φ), and

$\exists!\ Y_v \to Z$ s.t. $\rho(Y_v/Z) = 1$, Y_v $\mathbb{Q}$-fact. v div on Y.

Pf: BCHM.

Thm 1.9 (DCC of vol for bir bounded family)

Fix $I \subseteq [0,1]$ with DCC. $\mathfrak{D}$ a set of SNC pairs (X,Δ) which is log bir bounded, and $\mathrm{coeff}(\Delta) \subseteq I$. Then $\{\mathrm{vol}(K_X+\Delta) : (X,\Delta) \in \mathfrak{D}\}$ has DCC.

Proof

Step 0 Reduction.

Log bir bdd: $\exists\, (Z,B) \to T$, B reduced, s.t. $\forall (X,\Delta) \in \mathfrak{D}$

$\exists\, t \in T$, $X \overset{f}{\dashrightarrow} Z_t$, $\mathrm{supp}(B_t) \supseteq \mathrm{supp}(f_*\Delta) \cup \mathrm{exc}(f^{-1})$

Assume $1 \in I$. Can replace $(Z,\Phi) \leftarrow (Z', \Phi')$

$(X,\Delta) \leftarrow (X', M_{\Delta,X'})$

Assume:

- T reduced connected.
- (Z,B) is SNC over T.
- Every stratum of (Z,B) irred fiber over T
- T smooth.

Step 1 Reduce to $T = pt$, using Thm 1.8.

$$(X, \Delta) \in \mathfrak{D}, \quad X \xrightarrow{f} Z_t, \quad \Phi = f_* \Delta.$$

Strategy: Find Ψ_t on Z_t with $\mathrm{coeff}(\Psi_t) \subseteq I$

and $\mathrm{vol}(X, \Delta) = \mathrm{vol}(Z_t, \Psi_t) \underset{\substack{\uparrow \\ (\text{Thm } 1.8)}}{=} \mathrm{vol}(Z_0, \Psi_0)$.

Let $\Sigma = \{ v : v \text{ is a div on } X \text{ and } f\text{-exc}, \ \mathcal{L}_{\Phi}(v) > 0 \}$.

Then $\exists$ blow-up of strata of (Z_t, Φ): $X' \xrightarrow{f'} Z_t$

s.t. $\forall v \in \Sigma$ is a div on X'.

$$\begin{tikzcd}
& W \arrow[dl, "g"'] \arrow[dr, "g'"] & \\
X \arrow[rr, dashed] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& Z_t &
\end{tikzcd}$$

Let $\Delta' = \mathcal{M}_{\Delta, X'}$ ($\mathrm{coeff}(\Delta) \subseteq I$).

Claim: $\mathrm{vol}(K_X + \Delta) = \mathrm{vol}(K_{X'} + \Delta')$.

(Lem 5.3)

Assume: f blows up strata (Z_t, Φ).

(Z, B) SNC $/T$. Assume (strata B) $\cap Z_t$ = strata B_t.

$\exists$ Blow up $Z' \to Z$, and div B' on Z' s.t.

$$X' = Z_t'', \quad \Delta' = B_t'.$$

$$\mathrm{vol}(K_X + \Delta) = \mathrm{vol}(K_{X'} + \Delta') = \mathrm{vol}(Z_t', B_t')$$
$$\overset{(1.8)}{=} \mathrm{vol}(Z_0', B_0').$$

<u>Step 2</u> Suppose $(X_i, \Delta_i) \xrightarrow{f_i} Z$, $\mathrm{coeff}(\Delta_i) \subset I$

$$v_i = \mathrm{vol}(K_{X_i} + \Delta_i), \quad f_{i*}\Delta_i = \Phi_i \leq B.$$

Assume:
- $V_i \geq V_{i+1}$, $V = \lim V_i$
- For fixed val v, $\mathcal{M}_{\Delta_i}(v)$ is nondecreasing.
- For any val v

<u>Goal</u>: $V = V_i$.

⌈Strategy: $\Phi = \lim \Phi_i \geq \Phi_i$.

$$\mathrm{vol}(K_{X_i} + \Delta_i) \leq \mathrm{vol}(Z, K_Z + \Phi_i) \leq \mathrm{vol}(Z, K_Z + \Phi),$$

It suffices: $\forall \varepsilon > 0$, for suff. large i.

$$\mathrm{vol}(K_{X_i} + \Delta_i) \geq \mathrm{vol}(Z, K_Z + (1-\varepsilon)\Phi)$$
$$(\Rightarrow \ V \geq \mathrm{vol}(K_Z + \Phi))$$

Suppose $(Y, \overline{\Psi} = \mathcal{L}_{(1-\varepsilon)\overline{\Phi}, Y}) \to Z$ terminal SNC model.

$$\operatorname{vol}(Z, K_Z + (1-\varepsilon)\overline{\Phi}) \overset{\text{Lem}}{=} \operatorname{vol}(K_Y + \overline{\Psi})$$

$$= \operatorname{vol}(K_W + \mathcal{L}_{\overline{\Psi}, W})$$

$$\leq \operatorname{vol}(K_{X_i} + \mathcal{L}_{\overline{\Psi}, X_i})$$

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X_i \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

Want: $\Delta_i \geq \underset{\substack{\| \\ \text{bir transf. of } \overline{\Psi} \text{ on } X_i}}{\mathcal{L}_{\overline{\Psi}, X_i}}$.

$$\Longleftrightarrow \mathcal{M}_{\Delta_i, Y} \geq \overline{\Psi}$$

<u>Step 3</u>. $\mathcal{B} = \lim \mathcal{M}_{\Delta_i}$, $\mathcal{B}_Z = \overline{\Phi} = \lim \operatorname{fix} \Delta_i$.

Case I: $\mathcal{L}_{\overline{\Phi}} \leq \mathcal{B}$.

Pick $(Y, \overline{\Psi} = \mathcal{L}_{(1-\varepsilon)\overline{\Phi}, Y})$ termal SNC model.

For small $\varepsilon > 0$, $\exists\, \eta(\varepsilon) > 0$ s.t.

$$\mathcal{L}_{(1-\varepsilon)\overline{\Phi}, Y} \leq (1-\eta)\mathcal{L}_{\overline{\Phi}, Y} < \mathcal{L}_{\overline{\Phi}, Y} \leq \mathcal{B}_Y = \lim \mathcal{M}_{\Delta_i, Y}.$$

$\Longrightarrow$ $\cancel{\forall}$ For $i \gg 1$, $\mathcal{L}_{(1-\varepsilon)\overline{\Phi}, Y} \leq \mathcal{M}_{\Delta_i, Y}$, so done!

Case II: $\mathcal{L}_\Phi > \mathcal{B}$

Goal: Replace $(Z, \Phi) \longleftarrow (Z', \Phi')$

$$(X_i', \Delta_i') \longrightarrow Z',$$

$\mathcal{B}$ by $\mathcal{B}'$ on Z' s.t.

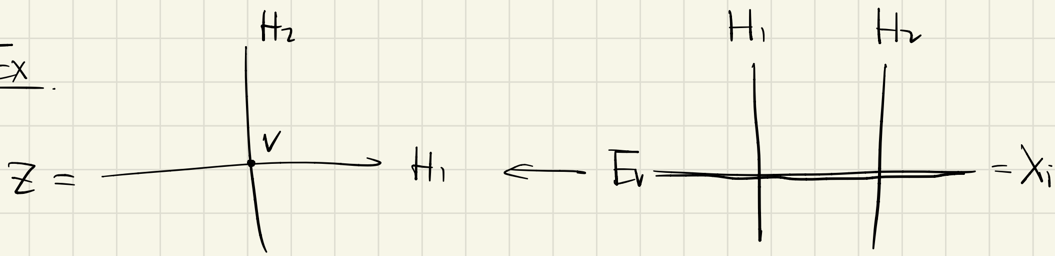
(1) $\operatorname{vol}(K_{X_i} + \Delta_i) = \operatorname{vol}(K_{X_i'} + \Delta_i')$

(2) $I' = \bigcup \operatorname{coeff}(\Delta_i')$ DCC

(3) $\mathcal{B}' = \lim \mathcal{M}_{\Delta_i'}$, $\Phi' = \mathcal{B}'_{Z'}$.

(4) $\boxed{\mathcal{L}_{\Phi'} \leq \mathcal{B}'}$

Ex.



$\Phi = H_1 + H_2$ $\qquad\qquad$ $\Delta_i = H_1 + H_2 + c_i E_v$

v: blow up of $H_1 \cap H_2$

$\mathcal{L}_\Phi(v) = 1$ $\qquad\qquad$ $\mathcal{B}(v) = \lim \mathcal{M}_{\Delta_i}(v)$

$$= \lim c_i < 1.$$

Solution: should replace Z by $\mathrm{Bl}_v Z$!

<u>Def</u> $\mathcal{B}$ b-div on Z, (Z, Φ) SNC.

For $Z' \xrightarrow{f} (Z, \Phi)$ log resol, define $(Z', \mathcal{B}', \Phi')$ to be:

- $\Sigma = \{\sigma : \sigma$ is f-exc div on Z', $\mathcal{L}_\Phi(\sigma) > 0\}$

- $\mathcal{B}' := \mathcal{B} \wedge \mathcal{M}_\Theta$, where Θ on Z'

$$\Theta = \bigwedge_{\sigma \in \Sigma} \mathcal{L}_{T_\sigma, Z'}, \quad T_\sigma = (\mathcal{L}_\Phi \wedge \mathcal{B})_{Y_\sigma}. \quad (Y_\sigma \to Z \text{ extracts } \sigma),$$

<u>Rmk</u>: $\Phi' = \mathcal{B}'_{Z'} \leq \mathcal{L}_{\Phi, Z'}$.

<u>Lemma 5.7</u> $\forall (Z, \mathcal{B}, \Phi)$ SNC, $\exists$ seq. of processes above.

ending with $(Z', \mathcal{B}', \Phi')$ s.t.

$$\mathcal{L}_{\Phi'} \leq \mathcal{B}'.$$

<u>Pf</u>. For each stratum W of Φ:

$$w = \begin{cases} \#\{D : \mathrm{coeff}_D(\Phi) = 1,\ D \supseteq W\}, & \text{if } \exists v \text{ s.t. } \mathrm{Cent}(v) = W,\ \mathcal{B}(v) < \mathcal{L}_\Phi(v) \\ -1, & \text{else.} \end{cases}$$

Goal: Find $(Z', \mathcal{B}')$ that decreases w.

Assume : $Z = \mathbb{A}^n$, $\Phi = \sum c_i H_i$. $0 \le c_1 \le \cdots \le c_n \le 1$.

$c_s < 1$, $c_{s+1} = 1$

where $s = n - w$.

Then every toric val $v = (v_1, \cdots, v_n) \in \mathbb{N}^n$.

$$\mathcal{L}_\Phi(v) = 1 - \sum_{i=1}^{s} v_i (1 - c_i) (> 0)$$

$$\mathcal{F} = \{ (v_1, \cdots, v_s) : 1 - \sum_{i=1}^{s} v_i (1 - c_i) > 0 \} \subseteq \mathbb{N}^s.$$

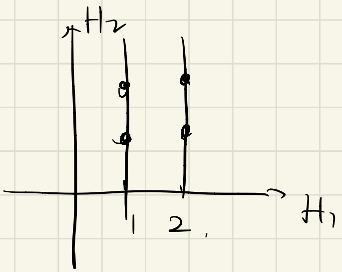
$\forall (v_1, \cdots, v_s) \in \mathcal{F}$, find $v_{s+1}, \cdots, v_n$ s.t.

$$\mathcal{B}(v_1, \cdots, v_n) = \min_{u_i \in \mathbb{N}} \{ \mathcal{B}(v_1, \cdots, v_s, u_{s+1}, \cdots, u_n) \}.$$

$\uparrow$

Σ set val. $\quad \Sigma \longleftrightarrow \mathcal{F}$.

<u>Ex.</u> $\mathbb{A}^2$, $\frac{2}{3} H_1 + H_2$



Let $Z' \longrightarrow (Z, \Phi)$ log resol s.t. $\forall \sigma \in \Sigma$,

σ is a div on Z', and $Z' \longrightarrow Y_\sigma$

(Z', B', Φ') be defined as in the def.

<u>Claim</u>: $\mathrm{wt}(Z', \Phi') < W$.

(1) $W = 0$.

If $\mathcal{L}_{\Phi'}(v) > B'(v) \geq 0$.

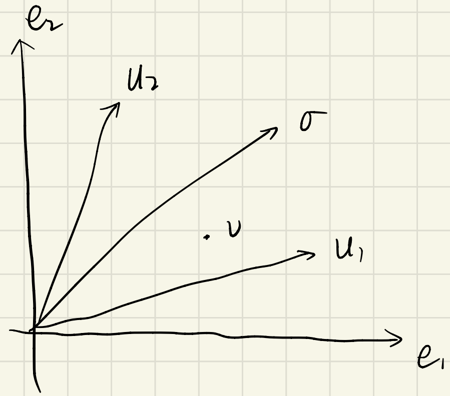
$$\mathcal{L}_{\Phi'}(v) > 0 \Rightarrow \mathcal{L}_{\Phi}(v) \geq \mathcal{L}_{\Phi'}(v) > 0.$$

$$\Rightarrow v \in \Sigma$$

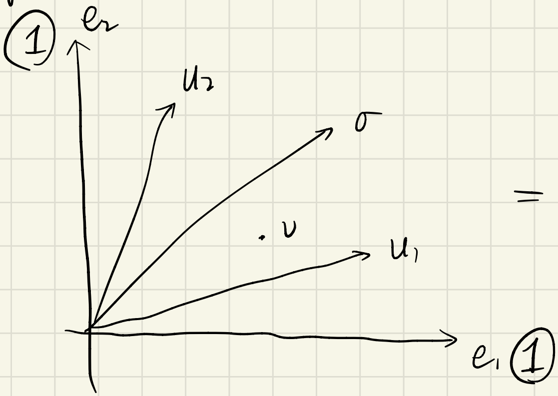
But then $\mathcal{L}_{\Phi'}(v) = B'(v) = \mathrm{coeff}_v(\Phi')$, contradiction!

(2) $W > 0$.

Proof by Pic:



Proof by Pic: Suppose $\mathrm{wt}(Z', \Phi') \geq w$.



$= (Z', \Phi')$.

$\mathcal{L}_\Phi(u_1) = 1 - \sum v_i(1-c_i)$

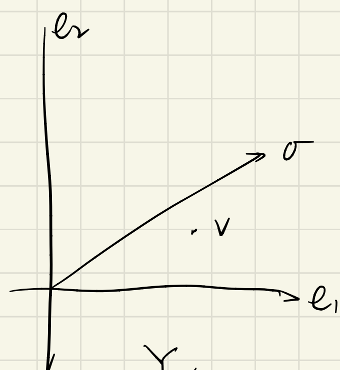
Suppose $\mathcal{L}_{\Phi'}(v) > \overset{\geq 0}{B'(v)}$, then $v = (v_1, \ldots, v_n)$

$\sigma \in \Sigma$: $\sigma = (v_1, \ldots, v_s, \ldots)$, $B(\sigma) \leq B(v) < 1$.

<u>Claim:</u> $\mathrm{wt}(Y_\sigma, T_\sigma = (\mathcal{L}_\Phi \wedge B)_{Y_\sigma}) \geq w$.

$\mathrm{wt}(Z', \Phi')$ at center(v)

$= \#\{u : u$ is in the subcone containing v and $u \in \Sigma\}$,



Y_σ

In part, v is in subcone spanned by

$\sigma, e_1, \ldots, \hat{e_l}, \ldots, e_n$, $l \leq s$.

$$v = \sum \lambda_i e_i + \lambda \sigma \qquad \sigma^i = v^i \text{ for } i \leq s.$$

$$\Longrightarrow \lambda = 1, \quad v \geqslant \sigma.$$

$$\begin{aligned} \mathcal{L}\Phi'(v) &\leqslant \mathcal{L}_{\mathbb{P}_\sigma}(v) \\ &\leqslant \mathcal{L}_{\mathbb{P}_\sigma}(\sigma) \\ &\leqslant \mathcal{B}(\sigma) \\ &\leqslant \mathcal{B}(v) \\ &= \mathcal{B}'(v). \end{aligned}$$

, contradiction!

$\boxtimes$